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Saint-Venant's problem for inhomogeneous and anisotropic solids

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SUMMARY
The present paper is concerned with Saint-Venant's problem for inhomogeneous and anisotropic elastic cylinders when the elastic coefficients are independent of the axial coordinate. The paper points out the importance of the generalized plane strain problem in the treatment of Saint-Venant's problem.

1. Introduction

Most of the papers concerned with Saint-Venant's problem are restricted to homogeneous or piecewise homogeneous elastic cylinders. However, some investigations (see e.g. [1–3]) are devoted to Saint-Venant's problem for inhomogeneous cylinders when the elastic coefficients are independent of the axial coordinate, and being prescribed functions of the remaining coordinates. In this case, even within the theory of isotropic elastic solids, the problem is difficult and it was entirely solved only when the Poisson's ratio is constant.

In this paper we give a method to solve Saint-Venant's problem for inhomogeneous and anisotropic elastic cylinders which avoids restrictions of this type. This method is essentially different from that of previous authors. The present paper points out the importance of the generalized plane strain problem in the treatment of Saint-Venant's problem. It is shown that the well known torsion function derives from a special problem of generalized plane strain.

2. Statement of the problem

Throughout this paper a rectangular coordinate system Ox_k $(k=1, 2, 3)$ is used. We consider a cylindrical body of inhomogeneous and anisotropic elastic material which occupies the region V of space, whose boundary is S. We suppose that the considered cylinder is bounded by plane ends perpendicular to the generators. We denote by L the boundary of the generic cross-section Σ. Throughout this paper the axis Ox_3 of our coordinate system will be directed parallel to the generators of the cylinder. The cylinder is assumed to be of length l, and one of its bases is taken to lie in the x_1Ox_2-plane, while the other is in the plane $x_3=l$. We shall employ the usual summation and differentiation conventions: Greek subscripts are understood to range over the integers (1, 2) whereas Latin subscripts—unless otherwise specified—are confined to the range (1, 2, 3); summation over repeated subscripts is implied and subscripts preceded by a comma denote partial differentiation with respect to the corresponding Cartesian coordinate. The linear theory of classical elasticity is considered.

Let u_i denote the components of the displacement vector field. Then the components of the infinitesimal strain field are given by

$$e_{ij}=\tfrac{1}{2}(u_{i,j}+u_{j,i})\,. \tag{2.1}$$

The stress-strain relations in the case of an anisotropic elastic medium are

$$t_{ij}=C_{ijkl}e_{kl}\,, \tag{2.2}$$

where t_{ij} are components of the stress tensor and C_{ijkr} are the components of the elasticity tensor which obey the symmetry relations

$$C_{ijkl} = C_{jikl} = C_{klij} . \tag{2.3}$$

We assume that the elasticity tensor is positive definite on V. Taking into account (2.1) and (2.3) the relations (2.2) can be written in the form

$$t_{ij} = C_{ijkl} u_{k,l} . \tag{2.4}$$

In this paper we consider an inhomogeneous medium for which we have

$$C_{ijkl} = C_{ijkl}(x_1, x_2) . \tag{2.5}$$

We assume that the domain Σ is C^∞-smooth [4] and the functions C_{ijkl} are supposed to belong to C^∞. We consider only a "C^∞-theory" but it is possible [4] to get a classical solution of the problem for more general domains and more general assumptions of regularity for the above functions (see e.g. [4], [5]). We have preferred this way in order to emphasize our method for solving the considered problem.

The equations of equilibrium, in the absence of body forces, are

$$t_{ij,j} = 0 . \tag{2.6}$$

The considered cylinder is supposed to be free from lateral loading, so that the conditions on the lateral surface are

$$t_{i\alpha} n_\alpha = 0 , \tag{2.7}$$

where $(n_1, n_2, 0)$ are the direction cosines of the exterior normal to lateral surface.

Saint-Venant's problem consists in the determination of the equilibrium of the considered cylinder which—in the absence of body forces—is subjected to surface tractions prescribed over its ends and is free from lateral loading. The treatment of this problem rests on a relaxed formulation in which the detailed assignment of the terminal tractions is abandoned in favour of prescribing merely the appropriate stress resultant. We assume that the load of the beam is distributed over its ends in a way which fulfills the equilibrium conditions of a rigid body. Let the loading applied on the end located at $x_3 = 0$ be statically equivalent to a force $P(P_i)$ and a moment $M(M_i)$. In what follows, for convenience, the problem of extension, bending and torsion is treated separately.

3. The generalized plain strain. Auxiliary plane strain problems

We define the state of generalized plane strain of the considered cylinder to be that state in which the components of the displacement vector depend only on x_1 and x_2

$$u_i = u_i(x_1, x_2) . \tag{3.1}$$

The above restriction implies that $e_{ij} = e_{ij}(x_1, x_2)$, $t_{ij} = t_{ij}(x_1, x_2)$. Further,

$$2e_{\alpha\beta} = u_{\alpha,\beta} + u_{\beta,\alpha} , \quad 2e_{\alpha 3} = 2e_{3\alpha} = u_{3,\alpha} , \quad e_{33} = 0 , \tag{3.2}$$

$$t_{i\alpha} = C_{i\alpha k\beta} u_{k,\beta} , \tag{3.3}$$

$$t_{33} = C_{33k\beta} u_{k,\beta} . \tag{3.4}$$

As a consequence of these relations, the equations of equilibrium, with the body forces f_i, take the form

$$t_{i\alpha,\alpha} + f_i = 0 , \tag{3.5}$$

from which it follows that the state of generalized plane strain demands that the components of body force be independent of x_3. Let us assume that on the lateral surface of the cylinder we have the boundary conditions

$$t_{i\alpha} n_\alpha = p_i \, . \tag{3.6}$$

Obviously the surface traction must be independent of x_3. The value of the stress t_{33} can be calculated after the displacement components are determined.

The generalized plane strain for homogeneous bodies was considered in various papers (see e.g. [6], [7]) under the assumption that $f_3 = p_3 = 0$. This restriction is superfluous. The question of the possibility of generalized plane strain in the considered elastic body will be answered in the affirmative sense even if f_3 and p_3 do not vanish.

The conditions of static equilibrium for the considered cylinder become

$$\begin{aligned} &\int_\Sigma f_i d\sigma + \int_L p_i ds = 0 \, , \quad \int_\Sigma \varepsilon_{\alpha\beta 3} x_\alpha f_\beta d\sigma + \int_L \varepsilon_{\alpha\beta 3} x_\alpha p_\beta ds = 0 \, , \\ &\int_\Sigma x_\alpha f_3 d\sigma + \int_L x_\alpha p_3 ds = \int_\Sigma t_{\alpha 3} d\sigma \, , \end{aligned} \tag{3.7}$$

where ε_{ijk} is the alternating symbol. The last two conditions (3.7) are identically satisfied on the basis of the relations (3.5), (3.6); thus

$$\begin{aligned} \int_\Sigma t_{\alpha 3} d\sigma &= \int_\Sigma [t_{\alpha 3} + x_\alpha (t_{3\beta,\beta} + f_3)] d\sigma = \int_\Sigma [(x_\alpha t_{3\beta})_{,\beta} + x_\alpha f_3] d\sigma = \\ &= \int_L x_\alpha p_3 ds + \int_\Sigma x_\alpha f_3 d\sigma \, . \end{aligned}$$

Let us study the existence of the solution of the considered problem. We introduce the operators

$$A_i u = -(C_{i\beta k\alpha} u_{k,\alpha})_{,\beta} \, , \tag{3.8}$$

where $u = (u_1, u_2, u_3)$. If we denote

$$f = (f_1, f_2, f_3) \, , \quad Au = (A_1 u, A_2 u, A_3 u) \, , \tag{3.9}$$

the equilibrium equations can be written in the form

$$Au = f \, , \text{ on } \Sigma \, . \tag{3.10}$$

Let $t(u)$ be the stress vector, with the components

$$t_i(u) = t_{i\alpha} n_\alpha = C_{i\alpha k\beta} u_{k,\beta} n_\alpha \, . \tag{3.11}$$

The boundary conditions (3.6) can be written in the form

$$t(u) = p \text{ on } L \, , \tag{3.12}$$

where $p = (p_1, p_2, p_3)$.

The functions f_i and p_i are supposed to belong to C^∞. Let u and v be two vectors of elastic displacements. Integrating by parts, we obtain

$$\int_\Sigma vAu d\sigma = 2\int_\Sigma W(u, v) d\sigma - \int_L vt(u) ds \, , \tag{3.13}$$

where

$$2W(u, v) = C_{i\alpha k\beta} e_{i\alpha}(u) e_{k\beta}(v) = C_{i\alpha k\beta} u_{i,\alpha} v_{k,\beta} \, , \tag{3.14}$$

is a bilinear form in the components of the deformation, corresponding to the quadratic form

$$2W(u) = C_{i\alpha k\beta} e_{i\alpha}(u) e_{k\beta}(u) \, , \tag{3.15}$$

which represents twice the potential energy density of the elastic deformation. This form is positive definite, so that

$$2W(u) \geqq \mu_0 e_{i\alpha} e_{i\alpha} \, , \quad \mu_0 > 0 \ (\mu_0 = \text{const.}) \, . \tag{3.16}$$

If u_i^0 is the solution of the homogeneous boundary value problem (3.10), (3.12), from (3.13), (3.16) we obtain

$$u^0_{\alpha,\beta} + u^0_{\beta,\alpha} = 0 \, , \quad u^0_{3,\alpha} = 0 \, ,$$

so that

$$u_\alpha^0 = a_\alpha + b\varepsilon_{\alpha\beta 3} x_\beta\,, \quad u_3^0 = a_3\,, \tag{3.17}$$

where a_i, b are arbitrary constants.

We consider the homogeneous boundary condition

$$t(u) = 0 \text{ on } L\,. \tag{3.18}$$

To prove the existence theorem for the boundary value problem (3.10), (3.18), as in [4, *p.* 381], we consider the system

$$Au + p_0 u = f\,, \tag{3.19}$$

where p_0 is an arbitrarily fixed positive constant. Firstly, we give an existence theorem for the boundary value problem (3.19), (3.18). Using (3.13), (3.15) it follows [4] that the inequality to be proven in this case is the following

$$\int_\Sigma e_{i\alpha} e_{i\alpha} d\sigma + \int_\Sigma u^2 d\sigma \geqq c_0 \|u\|_1^2\,, \quad c_0 > 0\,, \quad (c_0 = \text{const.})\,, \tag{3.20}$$

for any $u \in H_1(\Sigma)$. By $H_1(\Sigma)$ is denoted the Hilbert function space obtained by the functional completion of $C^1(\bar{\Sigma})$ with respect to the scalar product

$$(u, v)_1 = \int_\Sigma D^s u D^s v\, d\sigma\,, \quad (0 \leqq s \leqq 1)\,.$$

Using the second Korn's inequality [4], we can write

$$\int_\Sigma e_{\beta\alpha} e_{\beta\alpha} d\sigma + \int_\Sigma (u^{(1)})^2 d\sigma \geqq c_1 \|u^{(1)}\|_1^2\,, \quad c_1 > 0\,, \quad (c_1 = \text{const.})\,, \tag{3.21}$$

where $u^{(1)} = (u_1, u_2, 0)$. If we denote $u^{(2)} = (0, 0, u_3)$ we have

$$\int_\Sigma e_{3\alpha} e_{3\alpha} d\sigma + \int_\Sigma (u^{(2)})^2 d\sigma > \tfrac{1}{4} \|u^{(2)}\|_1^2\,. \tag{3.22}$$

From (3.21), (3.22) follows (3.20). Thus the boundary value problem (3.19), (3.18) has only one solution which is C^∞ in $\bar{\Sigma}$. The differential operator is formally self-adjoint, so that a C^∞ solution in $\bar{\Sigma}$ of the following system

$$Au + p_0 u - \lambda u = f\,, \tag{3.23}$$

with the boundary condition (3.18) exists if and only if

$$\int_\Sigma f u^* d\sigma = 0\,, \tag{3.24}$$

where u^* is any solution belonging to $C^\infty(\bar{\Sigma})$ of the problem (3.23), (3.18) with $f = 0$. In the case when $\lambda = p_0$ the only $C^\infty(\bar{\Sigma})$ solution of the homogeneous system is (3.17). Thus we have *Theorem 3.1.* The boundary value problem (3.10), (3.18) has solutions belonging to $C^\infty(\bar{\Sigma})$ if and only if the C^∞ vector f satisfies the conditions

$$\int_\Sigma f_i d\sigma = 0\,, \quad \int_\Sigma \varepsilon_{\alpha\beta 3} x_\alpha f_\beta d\sigma = 0\,. \tag{3.25}$$

Let us consider now the case of the inhomogeneous boundary condition. We assume that the C^∞ vector ψ satisfies the condition

$$t(\psi) = p \text{ on } L\,.$$

Let us introduce the vector w by the relation $w = u - \psi$. Then w satisfies the equation

$$Aw = f - A\psi\,, \tag{3.26}$$

and the homogeneous boundary condition (3.18). The necessary and sufficient conditions for the existence of the solution of the boundary value problem (3.26), (3.18) are

$$\int_\Sigma (f_i - A_i \psi) d\sigma = 0\,, \quad \int_\Sigma \varepsilon_{\alpha\beta 3} x_\alpha (f_\beta - A_\beta \psi) d\sigma = 0\,.$$

It is easy to show that

$$\int_\Sigma A_i \psi\, d\sigma = -\int_L p_i ds\,, \quad \int_\Sigma \varepsilon_{\alpha\beta 3} x_\alpha A_\beta \psi\, d\sigma = -\int_L \varepsilon_{\alpha\beta 3} x_\alpha p_\beta ds\,.$$

Thus, the necessary and sufficient conditions for the existence of the solution of the generalized plane strain problem are

$$\int_\Sigma f_i d\sigma + \int_L p_i ds = 0\,, \quad \int_\Sigma \varepsilon_{\alpha\beta 3} x_\alpha f_\beta d\sigma + \int_L \varepsilon_{\alpha\beta 3} x_\alpha p_\beta ds = 0\,. \tag{3.27}$$

The existence theorems in the case of homogeneous solids were established in [8].

We will have the opportunity to use four special problems, $P^{(s)}$ $(s=1, 2, 3, 4)$, of generalized plane strain. In what follows we denote by $v_i^{(s)}$, $\sigma_{i\alpha}^{(s)}$ the components of the displacement vector and the components of the stress tensor from the problem $P^{(s)}$. The problems $P^{(s)}$ are characterized by the equations

$$\sigma_{i\alpha}^{(s)} = C_{i\alpha k\beta} v_{k,\beta}^{(s)}\,, \tag{3.28}$$

$$\sigma_{i\alpha,\alpha}^{(\beta)} + (C_{i\alpha 33} x_\beta)_{,\alpha} = 0, \tag{3.29}$$

$$\sigma_{i\alpha,\alpha}^{(3)} + C_{i\alpha 33,\alpha} = 0\,, \tag{3.30}$$

and the boundary conditions

$$\sigma_{i\alpha,\alpha}^{(4)} - \varepsilon_{\rho\beta 3}(C_{i\alpha\rho 3} x_\beta)_{,\alpha} = 0 \ \text{ on } \ \Sigma\,, \tag{3.31}$$

$$\sigma_{i\alpha}^{(\beta)} n_\alpha = -C_{i\alpha 33} x_\beta n_\alpha\,, \tag{3.32}$$

$$\sigma_{i\alpha}^{(3)} n_\alpha = -C_{i\alpha 33} n_\alpha\,, \tag{3.33}$$

$$\sigma_{i\alpha}^{(4)} n_\alpha = \varepsilon_{\rho\beta 3} C_{i\alpha\rho 3} x_\beta n_\alpha \ \text{ on } \ L\,. \tag{3.34}$$

It is easy to show that the necessary and sufficient conditions (3.27) for the existence of the solution of the problem $P^{(s)}$ are satisfied. In what follows we assume that the functions $v_i^{(s)}$ and $\sigma_{i\alpha}^{(s)}$ are known.

When the material is isotropic, then

$$C_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})\,, \tag{3.35}$$

where λ and μ are the Lamé moduli. It is easy to see that for homogeneous and isotropic solids the solutions of the problems $P^{(s)}$ are

$$\begin{aligned}
&v_1^{(1)} = -\frac{\lambda}{4(\lambda+\mu)}(x_1^2 - x_2^2)\,, \quad v_2^{(1)} = -\frac{\lambda}{2(\lambda+\mu)} x_1 x_2\,, \quad v_3^{(1)} = 0\,,\\
&v_1^{(2)} = -\frac{\lambda}{2(\lambda+\mu)} x_1 x_2\,, \quad v_2^{(2)} = \frac{\lambda}{4(\lambda+\mu)}(x_1^2 - x_2^2)\,, \quad v_3^{(2)} = 0\,,\\
&v_\alpha^{(3)} = -\frac{\lambda}{2(\lambda+\mu)} x_\alpha\,, \quad v_3^{(3)} = 0\,, \quad v_\alpha^{(4)} = 0\,, \quad v_3^{(4)} = \varphi(x_1, x_2)\,,
\end{aligned} \tag{3.36}$$

where φ is the solution of the boundary value problem

$$\varphi_{,\alpha\alpha} = 0 \ \text{ on } \ \Sigma\,; \quad \varphi_{,\alpha} n_\alpha = \varepsilon_{\beta\alpha 3} x_\alpha n_\beta \ \text{ on } \ L\,. \tag{3.37}$$

4. Extension, bending and torsion

Let the loading applied on the end located at $x_3 = 0$ be statically equivalent to a force $P(0, 0, P_3)$ and a moment $M(M_1, M_2, M_3)$.

Thus, for $x_3 = 0$ we have the following conditions

$$\int_\Sigma t_{\alpha 3} d\sigma = 0\,, \tag{4.1}$$

$$\int_\Sigma t_{33} d\sigma = -P_3\,, \tag{4.2}$$

$$\int_\Sigma x_\alpha t_{33} d\sigma = \varepsilon_{\alpha\beta 3} M_\beta\,, \tag{4.3}$$

$$\int_\Sigma \varepsilon_{3\alpha\beta} x_\alpha t_{\beta 3} d\sigma = -M_3\,. \tag{4.4}$$

The resultant forces and moments calculated across each cross section satisfy the conditions of equilibrium, so that the conditions (4.1)–(4.4) must be satisfied for $x_3 = h$ $(0 \leq h \leq l)$.

The problem consists of solving the equations (2.4), (2.6) with the conditions (2.7), (4.1)–(4.4). We try to solve the problem assuming that

$$u_\alpha = -\tfrac{1}{2}a_\alpha x_3^2 - a_4 \varepsilon_{\alpha\beta 3} x_\beta x_3 + \sum_{s=1}^{4} a_s v_\alpha^{(s)} ,$$

$$u_3 = (a_1 x_1 + a_2 x_2 + a_3) x_3 + \sum_{s=1}^{4} a_s v_3^{(s)} , \tag{4.5}$$

where $v_i^{(s)}$ are the components of the displacement vector from the problem $P^{(s)}$ $(s = 1, 2, 3, 4)$ and a_s are unknown constants. From (4.5) we get

$$u_{k,\alpha} = a_\alpha x_3 \delta_{k3} - a_4 \varepsilon_{\beta\alpha 3} x_3 \delta_{k\beta} + \sum_{s=1}^{4} a_s v_{k,\alpha}^{(s)} ,$$

$$u_{k,3} = (a_1 x_1 + a_2 x_2 + a_3)\delta_{k3} - \delta_{k\alpha} a_\alpha x_3 - \delta_{k\alpha} a_4 \varepsilon_{\alpha\beta 3} x_\beta . \tag{4.6}$$

The components of the stress tensor have the form

$$t_{ij} = C_{ij33}(a_1 x_1 + a_2 x_2 + a_3) - C_{ij\alpha 3} \varepsilon_{\alpha\beta 3} a_4 x_\beta + \sum_{s=1}^{4} a_s \sigma_{ij}^{(s)} , \tag{4.7}$$

where $\sigma_{ij}^{(s)}$ are the components of the stress tensor from the problem $P^{(s)}$.

The equilibrium equations (2.6) and the boundary conditions (2.7) are satisfied on the basis of the relations (3.28)–(3.34). The conditions (4.1) are identically satisfied on the basis of the equilibrium equations and the boundary conditions (2.7); thus

$$\int_\Sigma t_{\alpha 3}\, d\sigma = \int_\Sigma (t_{\alpha 3} + x_\alpha t_{3\beta,\beta})\, d\sigma = \int_\Sigma (x_\alpha t_{3\beta})_{,\beta}\, d\sigma = \int_L x_\alpha t_{3\beta} n_\beta\, ds = 0 .$$

From (4.2)–(4.4) we obtain the following system for the unknown constants

$$\sum_{s=1}^{4} L_{\alpha s} a_s = \varepsilon_{\alpha\beta 3} M_\beta ,$$

$$\sum_{s=1}^{4} L_{3s} a_s = -P_3 , \quad \sum_{s=1}^{4} L_{4s} a_s = -M_3 , \tag{4.8}$$

where we used the notations

$$L_{\alpha\beta} = \int_\Sigma x_\alpha [C_{3333} x_\beta + \sigma_{33}^{(\beta)}]\, d\sigma ,$$
$$L_{\alpha 3} = \int_\Sigma x_\alpha [C_{3333} + \sigma_{33}^{(3)}]\, d\sigma ,$$
$$L_{\alpha 4} = \int_\Sigma x_\alpha [C_{33\rho 3} \varepsilon_{\beta\rho 3} x_\beta + \sigma_{33}^{(4)}]\, d\sigma ,$$
$$L_{3\alpha} = \int_\Sigma [C_{3333} x_\alpha + \sigma_{33}^{(\alpha)}]\, d\sigma ,$$
$$L_{33} = \int_\Sigma [C_{3333} + \sigma_{33}^{(3)}]\, d\sigma ,$$
$$L_{34} = \int_\Sigma [C_{33\alpha 3} \varepsilon_{\beta\alpha 3} x_\beta + \sigma_{33}^{(4)}]\, d\sigma ,$$
$$L_{41} = \int_\Sigma [C_{2333} x_1^2 - C_{1333} x_1 x_2 + \varepsilon_{3\alpha\beta} x_\alpha \sigma_{\beta 3}^{(1)}]\, d\sigma ,$$
$$L_{42} = \int_\Sigma [C_{2333} x_1 x_2 - C_{1333} x_2^2 + \varepsilon_{3\alpha\beta} x_\alpha \sigma_{\beta 3}^{(2)}]\, d\sigma ,$$
$$L_{43} = \int_\Sigma [C_{2333} x_1 - C_{1333} x_2 + \varepsilon_{3\alpha\beta} x_\alpha \sigma_{\beta 3}^{(3)}]\, d\sigma ,$$
$$L_{44} = \int_\Sigma [C_{2323} x_1^2 - 2C_{1323} x_1 x_2 + C_{1313} x_2^2 + \varepsilon_{3\alpha\beta} x_\alpha \sigma_{\beta 3}^{(4)}]\, d\sigma . \tag{4.9}$$

Let us prove that the system (4.8) determine uniquely the constants a_s. We assumed that the internal energy density

$$U(u) = \tfrac{1}{2}C_{ijkl}e_{ij}e_{kl}\,, \tag{4.10}$$

is a positive definite quadratic form.

Let us consider two elastic states $\{u'_i, e'_{ij}, t'_{ij}\}$ and $\{u''_i, e''_{ij}, t''_{ij}\}$. If we denote

$$2U(u', u'') = t'_{ij}e''_{ij}\,, \tag{4.11}$$

it follows that

$$U(u', u'') = U(u'', u')\,, \quad U(u, u) = U(u)\,. \tag{4.12}$$

Then Betti's formula leads to

$$2\textstyle\int_V U(u', u'')dv = \int_S t'_i u''_i d\sigma = \int_S t''_i u'_i d\sigma\,. \tag{4.13}$$

Obviously, we have

$$2\textstyle\int_V U(u)dv = \int_S t_i u_i d\sigma\,. \tag{4.14}$$

The relations (4.5), (4.7) can be written in the form

$$u_i = \sum_{s=1}^{4} a_s u_i^{(s)}\,, \quad t_{ij} = \sum_{s=1}^{4} a_s t_{ij}^{(s)}\,, \tag{4.15}$$

where

$$\begin{aligned} &u_\alpha^{(\beta)} = -\tfrac{1}{2}x_3^2\delta_{\alpha\beta} + v_\alpha^{(\beta)}\,, \quad u_3^{(\beta)} = x_3 x_\beta + v_3^{(\beta)}\,,\\ &u_i^{(3)} = \delta_{i3}x_3 + v_i^{(3)}\,, \quad u_\alpha^{(4)} = \varepsilon_{\beta\alpha 3}x_\beta x_3 + v_\alpha^{(4)}\,, \quad u_3^{(4)} = v_3^{(4)}\,,\\ &t_{ij}^{(\alpha)} = C_{ij33}x_\alpha + \sigma_{ij}^{(\alpha)}\,, \quad t_{ij}^{(3)} = C_{ij33} + \sigma_{ij}^{(3)}\,,\\ &t_{ij}^{(4)} = \sigma_{ij}^{(4)} - C_{ij\alpha 3}\varepsilon_{\alpha\beta 3}x_\beta\,. \end{aligned} \tag{4.16}$$

It is easy to show that

$$U(u) = \sum_{r,s=1}^{4} U_{rs}a_r a_s\,, \tag{4.17}$$

where

$$U_{rs} = U(u^{(r)}, u^{(s)}) = U_{sr}\,, \qquad (r, s = 1, 2, 3, 4)\,. \tag{4.18}$$

The total energy is

$$\mathscr{E} = \textstyle\int_V U(u)dv = \displaystyle\sum_{r,s=1}^{4} E_{rs}a_r a_s\,, \tag{4.19}$$

where

$$E_{rs} = \textstyle\int_V U_{rs}dv\,. \tag{4.20}$$

We note that

$$t_{i\alpha,\alpha}^{(s)} = 0 \text{ on } \Sigma\,, \quad t_{i\alpha}^{(s)}n_\alpha = 0 \text{ on } L\,, \quad (s = 1, 2, 3, 4)\,. \tag{4.21}$$

In view of these relations we get

$$\textstyle\int_\Sigma t_{\alpha 3}^{(s)}d\sigma = 0\,, \qquad (s = 1, 2, 3, 4)\,. \tag{4.22}$$

Let us apply the relations (4.13), (4.14) to the elastic states $\{u_i^{(s)}, e_{ij}^{(s)}, t_{ij}^{(s)}\}$, $(s=1, 2, 3, 4)$. Using the expressions of $u_i^{(s)}$, $t_{ij}^{(s)}$ given by (4.16) and the relations (4.22) we obtain

$$2E_{rs} = lL_{rs}\,, \qquad (r, s = 1, 2, 3, 4)\,. \tag{4.23}$$

Thus, with the help of (4.17), (4.18), (4.23) we conclude that

$$L_{rs} = L_{sr}\,, \quad \det(L_{rs}) > 0\,, \tag{4.24}$$

so that the system (4.8) determines uniquely the constants a_s. The considered problem is solved. On the basis of (3.36) it is easy to see that for homogeneous and isotropic solids the constants

L_{rs} reduce to

$$L_{\alpha\beta} = E\int_\Sigma x_\alpha x_\beta d\sigma \equiv EI_{\alpha\beta}\,, \quad L_{\alpha 3} = EAx^0_\alpha\,, \quad L_{33} = EA\,, \quad L_{i4} = 0\,,$$
$$L_{44} = \mu\int_\Sigma (x_1^2 + x_2^2 + x_1\varphi_{,2} - x_2\varphi_{,1})d\sigma \equiv D\,, \tag{4.25}$$

where E is Young's modulus, A is the area of the cross section and x^0_α are the coordinates of the centroid of Σ. The constant L_{44} is known as "torsional rigidity", and in most texts it is designated by D. In this case the system (4.8) becomes

$$I_{\alpha\beta} a_\beta + Ax^0_\alpha a_3 = \frac{1}{E}\varepsilon_{\alpha\beta 3} M_\beta\,, \tag{4.26}$$
$$a_1 x^0_1 + a_2 x^0_2 + a_3 = -\frac{1}{EA} P_3\,,$$
$$Da_4 = -M_3\,.$$

The constant a_4 is often denoted by τ and the function φ is known as the torsion function.

5. Flexure

The same calculation as in the treatment of the complete problem is implied in order to solve the flexure problem. For this reason we shall assume that the loading applied on the end $x_3 = 0$ is statically equivalent to a force $P(P_i)$ and a moment $M(M_i)$.

Thus, for $x_3 = 0$ we have the conditions

$$\int_\Sigma t_{\alpha 3} d\sigma = -P_\alpha\,, \tag{5.1}$$
$$\int_\Sigma t_{33} d\sigma = -P_3\,, \tag{5.2}$$
$$\int_\Sigma x_\alpha t_{33} d\sigma = \varepsilon_{\alpha\beta 3} M_\beta\,, \tag{5.3}$$
$$\int_\Sigma \varepsilon_{3\alpha\beta} x_\alpha t_{\beta 3} d\sigma = -M_3\,. \tag{5.4}$$

On the end located in the plane $x_3 = l$ we have the conditions

$$\int_\Sigma t_{\alpha 3} d\sigma = -P_\alpha\,, \tag{5.5}$$
$$\int_\Sigma t_{33} d\sigma = -P_3\,, \tag{5.6}$$
$$\int_\Sigma x_\alpha t_{33} d\sigma = \varepsilon_{\alpha\beta 3} M_\beta - lP_\alpha\,, \tag{5.7}$$
$$\int_\Sigma \varepsilon_{3\alpha\beta} x_\alpha t_{\beta 3} d\sigma = -M_3\,. \tag{5.8}$$

The problem consists in the solving of the equations (2.4), (2.6) with the conditions (2.7), (5.1)–(5.8).

We try to solve the problem assuming that

$$u_\alpha = -\tfrac{1}{2}a_\alpha x_3^2 - a_4\varepsilon_{\alpha\beta 3} x_\beta x_3 - \tfrac{1}{6}b_\alpha x_3^3 - \tfrac{1}{2}b_4\varepsilon_{\alpha\beta 3} x_\beta x_3^2 + $$
$$+ \sum_{s=1}^{4} (a_s + x_3 b_s) v_\alpha^{(s)} + v_\alpha(x_1, x_2)\,,$$

$$u_3 = (a_1 x_1 + a_2 x_2 + a_3)x_3 + \tfrac{1}{2}(b_1 x_1 + b_2 x_2 + b_3)x_3^2 +$$
$$+ \sum_{s=1}^{4} (a_s + x_3 b_s) v_3^{(s)} + v_3(x_1, x_2)\,, \tag{5.9}$$

where $v_i^{(s)}$ are the components of the displacement vector from the problems $P^{(s)}$ $(s = 1, 2, 3, 4)$, v_i are unknown functions and a_r, b_r $(r = 1, 2, 3, 4)$ are unknown constants.

From (2.4) and (5.9) we obtain

$$t_{ij}=C_{ij33}(a_1x_1+a_2x_2+a_3)-C_{ij\alpha 3}\varepsilon_{\alpha\beta 3}a_4x_\beta+C_{ij33}(b_1x_1+b_2x_2+$$
$$+b_3)x_3-C_{ij\alpha 3}\varepsilon_{\alpha\beta 3}b_4x_\beta x_3+\sum_{s=1}^{4}(a_s+x_3b_s)\sigma_{ij}^{(s)}+s_{ij}+k_{ij}, \tag{5.10}$$

where

$$s_{ij}=C_{ijk\beta}v_{k,\beta}\,,\; k_{ij}=C_{ijk3}\sum_{s=1}^{4}b_sv_k^{(s)}\,. \tag{5.11}$$

The relations (5.10) can be written as

$$t_{ij}=\pi_{ij}^{(0)}+x_3\pi_{ij}^{(1)}\,, \tag{5.12}$$

where

$$\pi_{ij}^{(0)}=C_{ij33}(a_1x_1+a_2x_2+a_3)-C_{ij\alpha 3}\varepsilon_{\alpha\beta 3}a_4x_\beta+\sum_{s=1}^{4}a_s\sigma_{ij}^{(s)}+s_{ij}+k_{ij}\,,$$
$$\pi_{ij}^{(1)}=C_{ij33}(b_1x_1+b_2x_2+b_3)-C_{ij\alpha 3}\varepsilon_{\alpha\beta 3}b_4x_\beta+\sum_{s=1}^{4}b_s\sigma_{ij}^{(s)}\,. \tag{5.13}$$

The conditions (5.2)–(5.4) and (5.6)–(5.8) are equivalent with the following conditions

$$\int_\Sigma\pi_{33}^{(1)}d\sigma=0\,,\quad\int_\Sigma x_\alpha\pi_{33}^{(1)}d\sigma=-P_\alpha\,,\quad\int_\Sigma\varepsilon_{3\alpha\beta}x_\alpha\pi_{\beta 3}^{(1)}d\sigma=0\,, \tag{5.14}$$

$$\int_\Sigma\pi_{33}^{(0)}d\sigma=-P_3\,,\quad\int_\Sigma x_\alpha\pi_{33}^{(0)}d\sigma=\varepsilon_{\alpha\beta 3}M_\beta\,,$$
$$\int_\Sigma\varepsilon_{3\alpha\beta}x_\alpha\pi_{\beta 3}^{(0)}d\sigma=-M_3\,. \tag{5.15}$$

From (5.13) and (5.14) we obtain the following system for the unknown constants b_r

$$\sum_{s=1}^{4}L_{rs}b_s=-P_\beta\delta_{r\beta}\,,\qquad(r=1,2,3,4)\,, \tag{5.16}$$

where L_{rs} are given by (4.9) and satisfy (4.24). In what follows we assume that the constants b_r are known.

Consider now the equilibrium equations. From (2.6), (3.29)–(3.31), (5.10) we obtain

$$s_{i\alpha,\alpha}+F_i=0\ \text{ on }\ \Sigma\,, \tag{5.17}$$

where

$$F_i=\pi_{i3}^{(1)}+k_{i\alpha,\alpha}\,. \tag{5.18}$$

On the basis of the relations (3.32)–(3.34) the conditions on the lateral surface (2.7) reduce to

$$s_{i\alpha}n_\alpha=-k_{i\alpha}n_\alpha\text{ on }L\,. \tag{5.19}$$

We consider the generalized plane strain problem (5.11), (5.17), (5.19). It is easy to show that the necessary and sufficient conditions to solve this problem are satisfied on the basis of the relations (5.16), (4.22). In what follows we assume that the functions v_i, s_{ij} are known.

Let us consider now the conditions (5.15). From (5.13) and (5.15) we obtain the following system for the constants a_r

$$\sum_{s=1}^{4}L_{\alpha s}a_s=\varepsilon_{3\alpha\beta}M_\beta-\int_\Sigma x_\alpha S_{33}d\sigma\,,$$
$$\sum_{s=1}^{4}L_{3s}a_s=-P_3-\int_\Sigma S_{33}d\sigma\,,$$
$$\sum_{s=1}^{4}L_{4s}a_s=-M_3-\int_\Sigma\varepsilon_{3\alpha\beta}x_\alpha S_{\beta 3}d\sigma\,;\;S_{i3}=s_{i3}+k_{i3}\,. \tag{5.20}$$

The conditions (5.1) and (5.5) are identically satisfied on the basis of the relations (5.14); thus

$$\int_\Sigma t_{\alpha 3}\, d\sigma = \int_\Sigma (t_{\alpha 3} + x_\alpha t_{3i,i})\, d\sigma = \int_L x_\alpha t_{3\beta} n_\beta\, ds + \int_\Sigma x_\alpha t_{33,3}\, d\sigma =$$
$$= \int_\Sigma x_\alpha \pi_{33}^{(1)}\, d\sigma = -P_\alpha\,.$$

The considered problem is solved.

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